複数の離心率および傾斜角を有する中間軌道

An Intermediary Orbit with two Eccentricities and two Inclinations

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Abstract. We reexamined the research done by Le Verrier concerning the theory of the motion of Mercury, if there were mistakes in his treatment. We could confirm the existence of the famous excess advance in the longitude of the perihelion. Even though we leaned on the theory of motion and the observational data used by him, we found an excess advance in the longitude of the ascending node which could not appear in his case because of his assumption (Inoue, 1991 and 1993a).

Careful examinations for the Le Verrier's studies have made us to recognize the circumstances that the cause of the appearance of such excess advances should lie in the traditional treatments. That is to say, one includes in lucking unity some portions of perturbations in the starting elliptic orbit, without paying any attention to the fact that such a treatment modifies the elliptic orbit to a different one which is not always given by the force inversely proportional to the square of the distance.

In order to make clear our standpoint, we introduced an intermediary orbit with two eccentricities and gave perfectly the quantity of the excess advance in the longitude of the perihelion of Mercury with the aid of the additional disturbing force (Inoue, 1992 and 1994). This means that if one takes into consideration the effect of the disturbing force for the Le Verrier's starting orbit, the famous excess advance of the perihelion disappears completely.

The present study reveals that the analogous idea of an intermediary orbit with two inclinations enables us to understand the reason why the excess advance of the ascending node appears.

§1. Introduction.

Le Verrier established a theory of the motion of Mercury based on the meridian observations (1859). In order to examine the validity of the theory, he applied the established theory to the transit observations of the planet on the disk of the Sun, and then found inadmissible discrepancies between his theory and the utilized observational data. In order to compensate these gaps, he was obliged to modify the values of secular parts, as well as constant parts, of the orbital elements. He considered that the correction for the longitude of the ascending node is of no use and that the following three elements are the principal ones, the mean longitude, the eccentricity and the longitude of the perihelion.

Solving the system of normal equations, he found excess secular motions in the longitude of the perihelion and in the mean longitude.

We reexamined this problem using the theory of the motion of Le Verrier and the observational data used by him. There might be someone who criticizes us because of the reason that his theory and his data are too antiquated to discuss such a delicate problem. In order to well persuade such a person, we would present here some details about our treatment (Inoue, 1993a). Even though the situations are almost same, there exist some differences between his procedures and ours :

- ☆ He analyzed separately the transit observations of May and those of November. We achieved the same analysis in unifying the whole observations.
- ☆ We did not suppose the non-existence of a secular influence due to the longitude of the ascending node over the observed latitudes of Mercury.

The following table gives the equations of condition for the twenty one transit observations used by Le Verrier (1859).

Number	*Date*	*Second Contact*	*Third Contact*
1	November 3, 1697		$0.39\delta v_1 - 0.26\delta s_1 + 0.''45 = \delta_1$
2	November 9, 1723	$0.45\delta v_2 - 0.10\delta s_2 - 0.''86 = \delta_2$	
3	November 11, 1736	$0.28\delta v_3 - 0.37\delta s_3 + 0.''75 = \delta_3$	
4	November 11, 1736		$0.16\delta v_4 + 0.43\delta s_4 + 0.''13 = \delta_4$
5	November 5, 1743	$0.34\delta v_5 + 0.32\delta s_5 - 0.''01 = \delta_5$	
6	November 5, 1743		$0.42\delta v_6 - 0.20\delta s_6 + 0.''92 = \delta_6$
7	November 9, 1769	$0.44\delta v_7 - 0.15\delta s_7 + 0.''99 = \delta_7$	
8	November 12, 1782	$0.17\delta v_8 - 0.45\delta s_8 - 0.''92 = \delta_8$	
9	November 12, 1782		$0.03\delta v_9 + 0.46\delta s_9 + 0.''23 = \delta_9$
10	November 5, 1789	$0.38\delta v_{10} + 0.27\delta s_{10} + 1.''81 = \delta_{10}$	
11	November 5, 1789		$0.44\delta v_{11} - 0.15\delta s_{11} + 0.''97 = \delta_{11}$
12	November 9, 1802		$0.46\delta v_{12} + 0.10\delta s_{12} + 1.''47 = \delta_{12}$
13	November 9, 1848	$0.46\delta v_{13} - 0.01\delta s_{13} + 2.''27 = \delta_{13}$	
14	May 6, 1753		$0.77\delta v_{14} - 0.27\delta s_{14} + 12.''05 = \delta_{14}$
15	May 4, 1786	$0.45\delta v_{15} - 0.70\delta s_{15} + 4.''84 = \delta_{15}$	
16	May 4, 1786		$0.65\delta v_{16} + 0.47\delta s_{16} + 5.''11 = \delta_{16}$
17	May 7, 1799	$0.80\delta v_{17} + 0.16\delta s_{17} + 5.''65 = \delta_{17}$	
18	May 7, 1799		$0.69\delta v_{18} - 0.43\delta s_{18} + 3.''83 = \delta_{18}$
19	May 5, 1832	$0.61\delta v_{19} - 0.53\delta s_{19} + 0.''17 = \delta_{19}$	
20	May 5, 1832		$0.77\delta v_{20} + 0.28\delta s_{20} - 0.''58 = \delta_{20}$
21	May 8, 1845	$0.74\delta v_{21} + 0.34\delta s_{21} - 1.''03 = \delta_{21}$	

We solved the system of normal equations under the assumption that there is no necessity of correction for the inclination. Our interest is limited only in the secular parts of the solution which are given as follows :

$$\delta \Omega_{(s)} = 6.''40\,07 \text{ /century} , \quad (1)$$
$$\delta \varpi_{(s)} = 39.''81\,29 \text{ /century} , \quad (2)$$
$$\delta \ell_{(s)} = 10.''50\,87 \text{ /century} . \quad (3)$$

where the notation $\delta \Omega_{(s)}$ expresses the secular correction for the longitude of the ascending node, the notation $\delta \varpi_{(s)}$ expresses the secular correction for the longitude of the perihelion and the notation $\delta \ell_{(s)}$ expresses the secular correction for the mean longitude.

The corresponding Le Verrier's solution gives the following values :

$$\delta \Omega_{(s)} \equiv 00.''0 \text{ /century} , \quad (4)$$
$$\delta \varpi_{(s)} = 38.''3 \text{ /century} , \quad (5)$$
$$\delta \ell_{(s)} = 10.''39 \text{ /century} . \quad (6)$$

As we easily see in the following table of the residuals, there exist several observations of which the precisions are not high.

Number	*Ours* (1) (2) (3)	*Le Verrier's* (4) (5) (6)	*Ours* (7) (8) (9)
1	0.″533 336	0.″56	-0.″001 091
2	-1.″017 025	-1.″08	【-1.″329.119 】
3	0.″423 541	0.″54	0.″119 472
4	0.″150 492	-0.″06	0.″282 042
5	-0.″277 101	-0.″44	-0.″288 777
6	0.″450 074	0.″47	0.″201 113
7	0.″122 156	0.″18	0.″015 736
8	-1.″230 534	-1.″20	【-1.″220 806 】
9	0.″037 527	0.″02	0.″004 104
10	0.″664 858	0.″77	【 0.″601 008 】
11	-0.″197 221	-0.″10	-0.″202 998
12	-0.″084 095	0.″09	-0.″098 812
13	-0.″033 644	0.″25	0.″147 377
14	0.″134 286	0.″07	-0.″044 323
15	0.″161 426	0.″11	0.″126 494
16	-0.″090 923	-0.″06	-0.″011 242
17	0.″722 046	0.″71	【 0.″929 639 】
18	-0.″906 019	-0.″82	【-0.″859 772 】
19	-0.″117 236	0.″08	-0.″069 753
20	-0.″604 069	-0.″67	0.″029 297
21	0.″705 392	0.″60	【 1.″520 384 】

Because the residuals are large in the observations of the *Numbers* 2, 8, 10 ; 17, 18 and 21, we rejected these six ones and obtained the following result :

$$\delta\Omega_{(s)} = 16.''7234 \text{ /century} , \qquad (7)$$
$$\delta\varpi_{(s)} = 41.''3418 \text{ /century} , \qquad (8)$$
$$\delta\ell_{(s)} = 11.''4629 \text{ /century} . \qquad (9)$$

In the table, the residuals enclosed by 【 】 indicate the predictions for the six rejected observations.
The value for the element $\delta\varpi_{(s)}$ is close to the value given by researchers of our days (Morrison and Ward, 1975).

$$\delta\varpi_{(s)} = 41.''9 \text{ /century} . \qquad (10)$$

These situations enable us to conclude that the theory and the observational data used by Le Verrier keep without doubt the contemporary significance. That is to say, we are able to discuss successfully the advance problem in leaning upon these quantities.

§ 2. Inclusion of the Periodic Perturbations.

The solution of the two-body problem is given by the following expressions with the orbital elements a, e, i ; Ω, ϖ and χ :

$$u - e \sin u = M = \int_{t_0}^{t} n \, dt + \chi \quad ; \qquad (11)$$

$$n \equiv \sqrt{\mu / a^3} \quad ; \qquad (12)$$

$$\tan(f/2) = \sqrt{(1+e)/(1-e)} \tan(u/2) \quad ; \qquad (13)$$

$$\psi \equiv \varpi - \Omega + f \quad ; \qquad (14)$$

$$r = a(1 - e \cos u) \quad ; \qquad (15)$$

$$\phi = \Omega + \tan^{-1}(\cos i \tan \psi) \quad ; \qquad (16)$$

$$\theta = \sin^{-1}(\sin i \sin \psi) \quad ; \qquad (17)$$

where r, ϕ and θ denote the heliocentric polar coordinates of the planet. The quantities M, u and f express the mean anomaly, the eccentric anomaly and the true anomaly respectively. The quantity μ is equal to $G(m_{\odot}+m)$ where G is the constant of gravitation and $m_{\odot}+m$ is the sum of the masses of the Sun and Mercury.
According to the usual manner, Le Verrier expressed the equation of the centre f − M and the radius vector r under the forms of the trigonometric series with respect to the mean anomaly M (Brouwer et al., 1961) :

$$f-M=\{2e-e^3/4+5e^5/96+107e^7/4608+\cdots\cdots\}\sin M+$$
$$+\{5e^2/4-11e^4/24+17e^6/192+\cdots\cdots\}\sin 2M+$$
$$+\{13e^3/12-43e^5/64+95e^7/512+\cdots\cdots\}\sin 3M+$$
$$+\{103e^4/96-451e^6/480+\cdots\cdots\}\sin 4M+ \qquad (18)$$
$$+\{1097e^5/960-5957e^7/4608+\cdots\cdots\}\sin 5M+$$
$$+\{1223e^6/960+\cdots\cdots\}\sin 6M+$$
$$+\{47273e^7/32256+\cdots\cdots\}\sin 7M+\cdots\cdots \quad ;$$

$$r=a(1+e^2/2)+$$
$$+a\{-e+3e^3/8-5e^5/192+7e^7/9216+\cdots\cdots\}\cos M+$$
$$+a\{-e^2/2+e^4/3-e^6/16+\cdots\cdots\}\cos 2M+$$
$$+a\{-3e^3/8+45e^5/128-567e^7/5120+\cdots\cdots\}\cos 3M+$$
$$+a\{-e^4/3+2e^6/5+\cdots\cdots\}\cos 4M+ \qquad (19)$$
$$+a\{-125e^5/384+4375e^7/9216+\cdots\cdots\}\cos 5M+$$
$$+a\{-27e^6/80+\cdots\cdots\}\cos 6M+$$
$$+a\{-16807e^7/46080+\cdots\cdots\}\cos 7M+\cdots\cdots \quad .$$

Among the planetary perturbations on the motion of Mercury, there exist the terms which depend uniquely on the mean anomaly of the planet M :

$$\delta\psi^*= S\sin M+C\cos M \quad ; \qquad (20)$$
$$\delta r = S'\sin M+C'\cos M \quad ; \qquad (21)$$

where $\delta\psi^*$ and δr are respectively the planetary perturbations on the true longitude ψ^* which is equal to $\varpi+f$ and the radius vector r.

Then, the situation becomes complicated and one might easily confound these planetary perturbations with the elliptic orbit mentioned above. Indeed, if one changes the value of the eccentricity e by Δe and that of the longitude of the perihelion ϖ by $\Delta\varpi$, one obtains analogous expressions as follows.

$$\Delta\psi^*= 2\Delta e\sin M-2e\Delta\varpi\cos M \quad ; \qquad (22)$$
$$\Delta r=-ae\Delta\varpi\sin M-a\Delta e\cos M \quad . \qquad (23)$$

Le Verrier introduced a convenient treatment for these terms (1856). In summing up these corresponding terms, one comes to hand the following expressions.

$$d\psi^*\equiv\delta\psi^*+\Delta\psi^*=(S+2\Delta e)\sin M+(C-2e\Delta\varpi)\cos M \; ; \qquad (24)$$
$$dr\equiv\delta r+\Delta r=(S'-ae\Delta\varpi)\sin M+(C'-a\Delta e)\cos M \; . \qquad (25)$$

He proposed to choose the values of correction Δe and $\Delta\varpi$ as one can obtain the following relations.

$$d\psi^*= 0 \quad ; \qquad (26)$$
$$dr=(S'-aC/2)\sin M+(C'+aS/2)\cos M \quad . \qquad (27)$$

Because of the circumstances that the coefficients S, C, S' and C' are very small, it is well permissible to treat like that for an approximate theory.
For the concrete construction of the theory, Le Verrier did not treat this as mentioned above. It is better to listen to what he said (1859).

Il entre, dans l'expression des perturbations de la longitude, des termes dépendant uniquement de la longitude de Mercure même. On sait qu'on peut les négliger, pourvu qu'on ajoute au rayon certains termes dépendant du même argument (Chapitre VI, Tome II). Mais ces derniers termes sont insensibles, si ce n'est dans l'action de Vénus sur Mercure ; et, dans cette dernière théorie, ils se trouvent égaux et de signes contraires aux termes

$$-0.''011 \sin \ell \quad \text{et} \quad -0.''003 \cos \ell$$

provenant d'une autre source. Il résulte de ces considérations que les termes des perturbations qui dépendent uniquement de la longitude de Mercure peuvent être négligés soit dans la longitude, soit dans le rayon de Mercure(*).

(*) C'est ce qui a été fait en réduisant en Tables les inégalités produites dans le mouvement de Mercure par l'action de Vénus. En calculant relatives aux inégalités produites dans la longitude par les actions de la Terre et de Jupiter, on a, par mégarde, consevé les termes en sin ℓ et cos ℓ. Peu importe, puisque l'une et l'autre voie sont également légitimes.

In order to make apparently disappear the action due to Venus, he introduced an action caused by an indefinite perturbing source which is equal, as he mentioned above, to the quantities : $+0.''011 \sin \ell$ and $+0.''003 \cos \ell$; $\ell \equiv \varpi + M$.
But, he never neglected the action due to Venus. We should, then, understand his standpoit as he tried to include the influence of the action due to Venus into the quantities (the orbital elements) e and ϖ. That is to say, he expected the possibility of the existence of the following equations simultaneously :

$$S + 2\Delta e = 0 \quad , \quad C - 2e\Delta\varpi = 0 \quad ; \tag{28}$$

$$S' - a\,e\Delta\varpi = 0 \quad , \quad C' - a\,\Delta e = 0 \quad . \tag{29}$$

This is not legitimate in general because only one quantity Δe has to satisfy two equations at the same time. The situation is the same for the quantity $\Delta\varpi$.

$$\Delta e_M = -S/2 = -0.''032\ 6264 \quad ; \tag{30}$$

$$\Delta e_r = C'/a = -0.''029\ 4537 \quad ; \tag{31}$$

$$e\Delta\varpi_M = C/2 = 0.''000\ 12414 \quad ; \tag{32}$$

$$e\Delta\varpi_r = S'/a = 0.''000\ 19117 \quad . \tag{33}$$

In any case, we would like to trace him and see what happens. It is easy to see that the difference between the two quantities Δe_M and Δe_r is entirely small and one might take this discrepancy negligible. Therefore, it is not difficult to understand why Le Verrier applied such a procedure. Let us include separately thus obtained quantities Δe_M and Δe_r into the expressions of the equation of the centre $(f-M)$ and the radius vector r:

$$(f-M)^* \equiv \{2\Delta e_M + (2e - e^3/4 + 5e^5/96 + \cdots\cdots)\} \sin M + \\ + \{5e^2/4 - 11e^4/24 + \cdots\cdots\} \sin 2M + \\ + \{13e^3/12 - 43e^5/64 + \cdots\cdots\} \sin 3M + \cdots\cdots \quad ; \tag{34}$$

$$r^* \equiv a(1+e^2/2) + \\ + a\{-\Delta e_r + (-e + 3e^3/8 - 5e^5/192 + \cdots\cdots)\} \cos M + \\ + a\{-e^2/2 + e^4/3 - e^6/16 + \cdots\cdots\} \cos 2M + \cdots\cdots \quad . \tag{35}$$

Even though the difference between the quantities Δe_M and Δe_r is extremely small, it is an undeniable fact that the difference exists. This means that one can never obtain the expressions (34) and (35) from an elliptic orbit. In order to procure them, one should introduce different forces from the force inversely proportional to the square of the distance.

In the case of the latitude θ also, Le Verrier applied the same procedure as the true longitude ψ^* and the radius vector r. We should still listen to him (Le Verrier, 1856).

> *Les termes qui dépendront seulement de la longitude moyenne de la planète, pourront généralement être négligés, attendu qu'ils se confondent avec les termes analogues du mouvement elliptique.*

He said that one can neglect the planetary influence, in including it in the elliptic elements, if their arguments are equal to the mean anomaly (the mean longitude) of the elliptic motion of Mercury. We would like to discuss this poit in the next section.

§3. Spatial Intermediary Orbit.

So as to briefly express what Le Verrier did as mentioned above, we presented the idea of an intermediary orbit (Inoue, 1992 and 1994). This orbit having two eccentricities includes the principal periodic perturbation effects depending on the mean anomaly (the mean longitude) of Mercury.

For this time, we propose an intermediary orbit with two inclinations in adding to the two eccentricities which will enable us to understand the appearance of the excess secular variation in the longitude of the ascending node. The orbit is expressed by the quantities a, e, i ; Ω, ϖ and χ which play the analogous rôle as the orbital elements of an elliptic orbit.

$$u - e_1 \sin u = M = \int_{t_0}^{t} n \, dt + \chi \quad ; \qquad (36)$$

$$\tan (f/2) = \sqrt{\{(1+e_1)/(1-e_1)\}} \tan (u/2) \quad ; \qquad (37)$$

$$\psi \equiv \varpi - \Omega + f \quad ; \qquad (38)$$

$$r = a\,(1 - e_2 \cos u) \quad ; \qquad (39)$$

$$\phi = \Omega + \tan^{-1} (\cos i_1 \tan \psi) \quad ; \qquad (40)$$

$$\theta = \sin^{-1} (\sin i_2 \sin \psi) \quad . \qquad (41)$$

Here, we put as follows :

$$e_1 \equiv e + \Delta e_1 \quad , \quad e_2 \equiv e + \Delta e_2 \quad ; \qquad (42)$$

$$i_1 \equiv i + \Delta i_1 \quad , \quad i_2 \equiv i + \Delta i_2 \quad . \qquad (43)$$

The intermediary orbit differs very slightly from an elliptic orbit which is identical with the orbit given by the expressions (11), (12), (13), (14), (15), (16) and (17). We wright now this elliptic orbit as follows :

$$u_0 - e \sin u_0 = M = \int_{t_0}^{t} n \, dt + \chi \quad ; \qquad (44)$$

$$\tan (f_0/2) = \sqrt{\{(1+e)/(1-e)\}} \tan (u_0/2) \quad ; \qquad (45)$$

$$\psi_0 \equiv \varpi - \Omega + f_0 \quad ; \qquad (46)$$

$$r_0 = a\,(1 - e \cos u_0) \quad ; \qquad (47)$$

$$\phi_0 = \Omega + \tan^{-1} (\cos i \tan \psi_0) \quad ; \qquad (48)$$

$$\theta_0 = \sin^{-1} (\sin i \sin \psi_0) \quad . \qquad (49)$$

Because of the existence of the quantities Δe_1 , Δe_2 ; Δi_1 and Δi_2 , there appear differences between the intermediary orbit and the elliptic orbit. We put these differences under the following formes :

$$u \equiv u_0 + \Delta u \ , \quad f \equiv f_0 + \Delta f \ , \quad \psi \equiv \psi_0 + \Delta \psi \quad ; \qquad (50)$$

$$r \equiv r_0 + \Delta r \ , \quad \phi \equiv \phi_0 + \Delta \phi \ , \quad \theta \equiv \theta_0 + \Delta \theta \quad ; \qquad (51)$$

$$\dot{r} \equiv \dot{r}_0 + \Delta \dot{r} \ , \quad \dot{\phi} \equiv \dot{\phi}_0 + \Delta \dot{\phi} \ , \quad \dot{\theta} \equiv \dot{\theta}_0 + \Delta \dot{\theta} \quad . \qquad (52)$$

For the elliptic orbit, we have the energy integral as follows :

$$-\mu/(2a)=(1/2)\{(\dot{r}_0)^2+(r_0\cdot\cos\theta_0\cdot\dot{\phi}_0)^2+(r_0\cdot\dot{\theta}_0)^2\}-\mu/r_0 . \quad (53)$$

In utilizing the relations (51) and (52) and retaining only the first order of the differences, we are able to express this integral with the quantities of the intermediary orbit under the following form :

$$-\mu/(2a)=(1/2)\{\dot{r}^2+(r\cdot\cos\theta\cdot\dot{\phi})^2+(r\cdot\dot{\theta})^2\}-\{\mu/r+{}^{\epsilon}R\} ; \quad (54)$$

$${}^{\epsilon}R \equiv (\mu/p)(\Delta e/e\,\eta^2)\ \{-\eta^4+3\eta^2\xi-\xi^2-\xi^3\}+ +(\mu/p)(\Delta i/2)\ \xi^2\sin 2i\ \{\cos(2\phi-2\Omega)/\cos^2\theta\} ; \quad (55)$$

$$p\equiv a\eta^2,\quad \eta\equiv\sqrt{(1-e^2)},\quad \xi\equiv p/r=1+e\cos f ; \quad (56)$$

$$\Delta e\equiv e_2-e_1\equiv\Delta e_r-\Delta e_M=+3.''1726\times10^{-3}=+1.5381\times10^{-8} ; \quad (57)$$

$$\Delta i\equiv i_2-i_1= ? \quad . \quad (58)$$

If we put ${}^{\epsilon}R=0$, we will obtain the following form of the energy integral which corresponds to that of the two-body problem :

$$-\mu/(2a)=(1/2)\{\dot{r}^2+(r\cdot\cos\theta\cdot\dot{\phi})^2+(r\cdot\dot{\theta})^2\}-\mu/r \quad . \quad (59)$$

We may, then, take the quantity ${}^{\epsilon}R$ as the disturbing function for an elliptic orbit. It is evident that the function contains not only the coordinates but also the velocities. In this case, if we may take the constants a, e, i ; Ω, ϖ and χ as contact elements, it is not difficult to calculate the variations of these quantities through the ordinary formulae in the method of the variation of arbitrary constants (Brumberg, 1991). We should suppose the possibility of the correspondence : $\dot{r}\rightarrow p_r$, $(r\cdot\cos\theta)^2\cdot\dot{\phi}\rightarrow p_\phi$, $r^2\cdot\dot{\theta}\rightarrow p_\theta$. In place of the expression (59), we have a new expression as follows :

$$-\mu/(2a)=(1/2)\{p_r^2+(p_\phi/r\cdot\cos\theta)^2+(p_\theta/r)^2\}-\mu/r \quad . \quad (60)$$

After this, we were able to calculate the variations of the contact elements. Among them, the following secular ones are of interest :

$$\delta\varpi_{(s)} = -(\Delta e/e)\{2/(1+\eta)\}\{1/\eta+(e/\eta)^2\}\ f \quad ; \quad (61)$$

$$\delta\Omega_{(s)} = (63/4)\ \Delta i\ \sin i\ (\tan i/2)^4\ f \quad . \quad (62)$$

The evaluation of $\delta\varpi_{(s)}$ is straightforward because we know the value for the eccentricity e as 0.205 6105 (Le Verrier, 1859).

$$\delta\varpi_{(s)} = -0.''104\ 4651\ 84/\text{revolution} \quad . \quad (63)$$

We may also calculate this quantity for a century if we substitute the values 87.96 92580 days and 365.25 63744 days for the periods of revolution of Mercury and the Earth respectively (Le Verrier, 1856).

$$\delta \varpi_{(s)} = -43.''37 \text{ /century} \quad . \qquad (64)$$

This is very close to one of the actually current values (Duncombe, 1956).

$$\delta \varpi_{(s)} = +43.''11\text{/century} \quad . \qquad (65)$$

This means that the Le Verrier's starting orbit was almost elliptic but slowly retrograding with a magnitude of 43 arc-seconds per century (Inoue, 1992).

We now ask ourselves if we were able to compensate the value 16.″72 34/century of the excess advance in the longitude of the ascending node with the aid of the expression (62). The value for the inclination i is nearly equal to 7.°0. Then, the coefficient for the quantity $\Delta i \cdot f$ has only the magnitude of 10^{-5}. In this case, it is unfortunately inadmissible to equate these two quantities because the quantity Δi should take a value : $-238.''64$.

We would like to treat this problem differently. We introduce a quantity ϕ^* in place of the real longitude ϕ given by the following relation :

$$\phi^* \equiv \phi + \Delta i \tan i \; \{\phi - \sin(\phi - \Omega) \cos(\phi - \Omega)\} \quad . \qquad (66)$$

If we differentiate this relation with respect to the time, then we will obtain the following compact relation :

$$\dot{\phi}^* = (\cos i_1/\cos^2 \theta)\dot{\psi} \quad . \qquad (67)$$

The corresponding relations become as follows :

$$-\mu/(2a) = (1/2)\{\dot{r}^2 + (r \cdot \cos\theta \cdot \dot{\phi}^*)^2 + (r \cdot \dot{\theta})^2\} - \{\mu/r + {}^{\epsilon}R^*\} \; ; \qquad (54)^*$$

$$\begin{aligned} {}^{\epsilon}R^* \equiv\; & (\mu/p)(\Delta e/e\,\eta^2)\{-\eta^4 + 3\eta^2\xi - \xi^2 - \xi^3\} + \\ & + (\mu/p)(\Delta i/2)\; \xi^2 \sin 2i \; (1/\cos^2\theta) \end{aligned} \quad . \qquad (55)^*$$

The secular part of the longitude of the perihelion still remains unchanged. For the secular part of the longitude of the ascending node, we obtained newly the following approximated expression :

$$\delta \Omega_{(s)} = \Delta i \cot i \;\; f \quad . \qquad (68)$$

We calculated this quantity from the differential equation given below :

$$d\Omega/df = (\Delta i/\sin i)\; \{(-1/\cos^2\theta) + 2(\cos^2 i/\cos^4\theta)\} \quad . \qquad (69)$$

Using the value 7° 0′ 8.″16 for the orbital inclination i (Le Verrier, 1859), we can evaluate $\delta\Omega_{(s)}$ for a century. Let us put equal this quantity to the excess advance in the longitude of the ascending node given by (7) with the opposite sign in order to estimate the value of the quantity Δi :

$$21\,240.3\ \Delta i/\text{century} \equiv -16.''7\,234/\text{century} \quad ; \tag{70}$$

$$\Delta i = -7.''87\,342\times10^{-4} = -3.81\,714\times10^{-9} \quad . \tag{71}$$

This is a quantity surprisingly small but sufficient to make appear the excess advance of the order of 16 arc-seconds per century.

It is necessary to examine the variation of the orbital inclination because of the existence of Δi.

$$di/df = \Delta i\ \sin^2 i\ \cos i\ \{\sin(2\phi - 2\Omega)/\cos^2\theta\} \quad . \tag{72}$$

This is purely periodic and its amplitude does not exceed the following limit :

$$|di/df| \leqq |\Delta i|\ \sin i\ \tan i = 1.''17\,891\times10^{-5} \quad . \tag{73}$$

The existence of the quantity Δi influences the element ϖ also.

$$\begin{aligned} \delta\varpi_{(s)} &\equiv \{\delta\varpi_{(s)}\}_{\Delta i} + \{\delta\varpi_{(s)}\}_{\Delta e} = \\ &= -0.''12/\text{century} - 43.''37/\text{century} = \\ &= -43.''49/\text{century} \quad . \end{aligned} \tag{74}$$

These relations (73) and (74) assure us to introduce the latter quantity Δi as well as the former quantity Δe.

§ 4. Concluding Remarks.

The present study made clear that the introduction of the small quantity Δi enables us to compensate totally the excess secular variation in the longitude of the ascending node $\delta\Omega_{(s)}$. There might be an ambiguity for the evaluation of the quantity Δi because the value for the excess advance in the longitude of the ascending node +16.″72··· is not yet well authorized (Inoue, 1993b).

In the case of the quantity Δe, there is no ambiguity not at all. Because for the estimation, we are able to lean on the quantities S, C ; S' and C' which are definitely given by the planetary theory. With the aid of this quantity Δe, we could well understand the reason why there exists the excess advance in the longitude of the perihelion $\delta\varpi_{(s)}$, namely, there were overlooked portions in the theory of Le Verrier which changed the starting orbit from a fixed ellipse to a slowly retrograding one (Inoue, 1992 and 1994).

References

1.	Brouwer, D. & Clemence, G.M.	: 1961	*The Method of Celestial Mechanics*, 76.
2.	Brumberg, V.A.	: 1991,	*Essential Relativistic Celestial Mechanics*, 8.
3.	Duncombe, R.L.	: 1956,	*The Astronomical Journal*, 61.174.
4.	Inoue, T.	: 1991,	*Proceedings of the Twenty-Fourth Symposium on Celestial Mechanics* (held in Tokyo), 148.
5.	Inoue, T.	: 1992,	*Proceedings of the Twenty-Fifth Symposium on Celestial Mechanics* (held in Tokyo), 205.
6.	Inoue, T.	: 1993a,	*Acta Humanistica et Scientifica Universitatis Sangio Kyotiensis*, 23. 64.
7.	Inoue, T.	: 1993b,	*Celestial Mechanics and Dynamical Astronomy*, 56. 69.
8.	Inoue, T.	: 1994,	*Proceedings of the Twenty-Sixth Symposium on Celestial Mechanics* (held in Tokyo), 164.
9.	LeVerrier, U.J.	: 1856,	*Annales de l'Observatoire Impérial de Paris*, II. 39.
10.	LeVerrier, U.J.	: 1859,	*Annales de l'Observatoire Impérial de Paris*, V.
11.	Morrison, L.V. & Ward, C.G.	: 1975,	*Monthly Notices Roy. Astron. Soc.*, 173. 183.

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