第２６回天体力学研究会

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表　題　：　**『水星近日点前進問題の解決』への追補二三**
場　所　：　　国立天文台　（東京・三鷹）
期　日　：　１９９４年1月１２日（水）～１３日（木）

（１）　子午線観測に基づいた水星運動理論と其れに依る予報計算。
（２）　太陽面通過現象に於ける観測値と計算値の間の食い違い。
（３）　食い違い僅少化の為の条件方程式は『近日点黄経 ϖ 及び平均黄経 ℓ に【余剰の永年変化】の追加』を要求。

（一）　前回の「エネルギー積分」表式に対する修正。
（二）　■中間軌道■　を産み出すLagrange函数 $\mu/r+\varepsilon V$ の導出。「Lagrange の運動方程式の一性質」に依拠して「$\varepsilon V_1=\varepsilon R$」なる近似の採用。これに依る時の要素変化の式の計算と「エネルギー積分」の近似再現度の評価。
（三）　今一つのLagrange函数の摂動部分 εV_2 の近似表式の導出。
（四）　近日点黄経 ϖ に於ける余剰の永年変化も、平均黄経 ℓ に於ける余剰の永年変化も、前回発表の儘で総て正しい。

◎　惑星の運動理論樹立に際して《惑星の影響の一部を取り込む》と云う伝統的・常套的な手法は、初発の軌道を楕円軌道で無いものに改変して仕舞う。　■動く楕円軌道■　の闖入！
◎　此の方法に基づいて Le Verrier は［〔水星の運動理論〕］を樹立。楕円軌道に惑星の摂動を考慮。上で注意した「楕円軌道からのズレ」は考慮されては居らず、従って原因不明の【過不足】が生じたとしても怪しむに足らず。
◎　伝統的手法に起因して参入して来る初期軌道を、単純な表式で与えるのは容易ではない。我々はこれを「二個の離心率」e_M、e_r を有する軌道で近似。これを　■中間軌道■　と呼ぶ。
◎　此の　■中間軌道■、《楕円軌道にして　Δe の大きさの摂動作用を受けるが如きもの》に相当。（$\Delta e\equiv e_r-e_M$）。此の摂動に依って惹起される永年変化の量は条件方程式が要求する【過不足分】に完全に一致！

（☆）　近日点黄経 ϖ が、惑星摂動に依って変化する量は LeVerrier が計算した通り 527″/世紀で正しい。
（☆）　これを、■中間軌道■　採用の認識無しに計算に供すれば ϖ' = 527″/世紀 ＋(−43″/世紀) = 484″/世紀　は必然。
（☆）　以上から、近日点黄経に於ける（Ｏｂｓ．－Ｃａｌ．）は条件方程式が教えて呉れる通り　＋43″/世紀　となる。

[527″/世紀－ {527″/世紀＋(－43″/世紀)}] ＝ ＋43″/世紀

「水星近日点前進問題の解決」への追補二三

Additional remarks for the *"Termination of the problem of the excess advance in the longitude of the perihelion of Mercury"*

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Abstract. We correct mistakes contained in our intermediary orbit and then give the correct expression for *'the energy integral'* (Inoue, 1992). In defiance of the existence of the errors, the previously concluded results are never affected and remain wholly valid for *'the termination of the excess advance problem'*.

§ 1. Fundamental relations.

Let us begin with giving the expressions for the solution of the *planar* two-body problem by means of the orbital elements a, e ; ϖ and ε. In order to reinforce our standpoint, we do not hesitate to enumerate fundamental relations even the case where the relations were entirely elementary ones.

(1) $$u - e \sin u = M = \ell - \varpi \ ,$$

(2) $$\ell = \int_{t_0}^{t} n \, dt + \varepsilon \ , \quad \text{(mean longitude)} ,$$

(3) $$\tan (f/2) = \sqrt{\{(1+e)/(1-e)\}} \cdot \tan (u/2) \quad ;$$

(4) $$r = a \, (1 - e \cos u) \quad ,$$

(5) $$\phi = \varpi + f \ , \qquad \text{(true longitude)} .$$

(6) $$f = M + \{2e - e^3/4 + \cdots\} \sin M + \{5e^2/4 - \cdots\} \sin 2M + \cdots \ .$$

§ 2. LeVerrier's treatments.

Suppose that there exist errors Δe and $\Delta\varpi$ in the elements e and ϖ, then we obtain the following diffrential relations through the equation of the centre and the expression for the radius r.

(7) $$\Delta\phi = 2\Delta e \{1 - 3e^2/8 + \cdots\} \sin M + - 2e\Delta\varpi \{1 - e^2/8 + \cdots\} \cos M + \cdots ,$$

(8) $$\Delta r = - (a e \Delta\varpi/\eta) \sin f - a \Delta e \cos f \quad ;$$

(9) $$(\eta \equiv \sqrt{(1-e^2)} \ )$$

In neglecting the periodic parts of the order of square with respect to the eccentricity, LeVerrier (1856) approximated the above relations as follows.

(10) $$\Delta\phi = 2\,\Delta e \sin M - 2e\Delta\varpi \cos M \ ,$$

(11) $$\Delta r = -\ a\,e\Delta\varpi \sin M - a\Delta e \cos M \ .$$

Among the planetary perturbations, there exist the terms which depend uniquely on the mean anomaly of Mercury M .

(12) $$\delta\phi = S \sin M + C \cos M \ ,$$

(13) $$\delta r = S' \sin M + C' \cos M \ .$$

If the case is not that of Venus, namely of the Earth or Jupiter or any other else, the coefficients S, C ; S' and C' are very small. In view of these facts, LeVerrier (1859) made absorb the perturbed effects in the elements e and ϖ in compensating with the corrections Δe and $\Delta\varpi$.

(14) $$d\phi \equiv \delta\phi + \Delta\phi = 0 \ ,$$

(15) $$dr \equiv \delta r + \Delta r = 0 \ .$$

This means that there is a possibility to appear two different values as correction for each of the elements e and ϖ.

(16) $$\Delta e_1 = -S/2 \ , \qquad e\Delta\varpi_1 = C/2 \ , \ (\text{from } d\phi = 0) \ ;$$

(17) $$\Delta e_2 = C'/a \ , \qquad e\Delta\varpi_2 = S'/a \ , \ (\text{from } dr = 0) \ .$$

The corresponding numerical values are calculated through the data given in p. 13 of the Volume V.

(18) $$\Delta e_1 = -0.''032\ 6264 \ , \quad e\Delta\varpi_1 = 0.''000\ 1241 \ ;$$

(19) $$\Delta e_2 = -0.''029\ 4537 \ , \quad e\Delta\varpi_2 = 0.''000\ 1911 \ .$$

The corrections for the longitude of the perihelion is of no importance for the present studies, and then we are going to concentrate in the corrections for the eccentricity. The difference between the two quantities Δe_1 and Δe_2 is very small and one may take this discrepancy negligible. Is this right? Our interest is not to reply to this question, but to know the orbit which permits to contain such planetary effects as mentioned above.

§3. Intermediary orbit.

So as to make absorb the principal periodic perturbation effects especially due to Venus, which depend on the mean anomaly of Mercury, we proposed the idea of the intermediary orbit (Inoue, 1992). The intermediary orbit was not correct because of the existence of an error. The correct set of the expressions for the intermediary orbit is given with the constants of integration a, e ; ϖ and ε.

(20) $$u - e_M \sin u = M = \ell - \varpi \ ,$$

(21) $$\ell = \int_{t_0}^{t} n\,dt + \varepsilon, \quad n \equiv \sqrt{(\mu/a^3)} \quad ;$$

(22) $$\tan(f/2) = \sqrt{\{(1+e_M)/(1-e_M)\}} \cdot \tan(u/2) \quad ;$$

(23) $$r = a\,(1 - e_r \cos u) \quad ,$$

(24) $$\phi = \varpi + f \ .$$

We introduced here two quantities e_M and e_r given by the following relations.

(25) $$e_M = e + \Delta e_1 \ ,$$

(26) $$e_r = e + \Delta e_2 \ .$$

For this time, our present intermediary orbit gives indeed the desired differential relations for the true longitude ϕ and the radius r as follows.

(27) $$\Delta\phi = 2\,\Delta e_1 \cdot \sin M + 0 \quad ,$$

(28) $$\Delta r = 0 - a\Delta e_2 \cdot \cos M \ .$$

The corresponding *'energy integral'* has the following form that differs slightly from the previously given one. The portion ${}^{\epsilon}R$ plays not exactly but a sort of the role of the disturbing function for the problem of two bodies.

(29) $$-\mu/(2a) = \{\dot{r}^2 + (r\dot{\phi})^2\}/2 - \{\mu/r + {}^{\epsilon}R\} \quad ,$$

(30) $${}^{\epsilon}R \equiv (\mu\Delta e/e)\ \{-(a\eta)^2/r^3 - a/r^2 + 3/r - 1/a\} \quad ;$$

where we newly introduced the notation Δe as follows.

(31) $$\Delta e \equiv e_r - e_M = \Delta e_2 - \Delta e_1 \ .$$

§ 4. Lagrangian and perturbing forces.

We have to consider that the perturbing part of *'this energy integral'* ${}^{\epsilon}R$ depends on the velocities $\dot{r}$ and $r\dot{\phi}$ through the constants of integration a and e. In such a case, to find the disturbing function is not easy. In order to investigate this problem, let us introduce *the Lagrangian* L which enables us to produce *'the energy integral'*, in other words *'the Hamiltonian* H*'*.

(32) $$L = \{\dot{r}^2 + (r\dot{\phi})^2\}/2 + \mu/r + {}^{\epsilon}V(r, \phi; \dot{r}, \dot{\phi}; t) \quad ;$$

(33) $$H \equiv (\partial L/\partial \dot{r}) \cdot \dot{r} + (\partial L/\partial \dot{\phi}) \cdot \dot{\phi} - L =$$

$$= \{\dot{r}^2 + (r\dot{\phi})^2\}/2 - \{\mu/r + {}^{\epsilon}V + - (\partial\,{}^{\epsilon}V/\partial \dot{r}) \cdot \dot{r} - (\partial\,{}^{\epsilon}V/\partial \dot{\phi}) \cdot \dot{\phi}\} \ .$$

In comparing *the Hamiltonian* (33) with *'the energy integral'* (29), we obtain a partial differential equation for the unknown function ${}^{\epsilon}V$ as follows.

(34) $\quad -\mu/(2a) = H$;

(35) $\quad {}^{\epsilon}R = {}^{\epsilon}V - (\partial\, {}^{\epsilon}V/\partial \dot{r}) \cdot \dot{r} - (\partial\, {}^{\epsilon}V/\partial \dot{\phi}) \cdot \dot{\phi}$.

It seems troublesome to solve exactly the partial differential equation for the unknown function ${}^{\epsilon}V$. We will therefore satisfy ourselves by obtaining some approximated expressions for the function.

Before to enter in the discussion about the solution of this equation, we will touch on the method of the variation of the constants of integration, namely the elements a, e ; ϖ and ε. The equations of motion of Lagrange teach us how to find the components of the disturbing force. We express these components by the notations $\mathbb{R}$ and $\mathbb{T}$.

(36) $\quad d(\partial L/\partial \dot{r})/dt - \partial L/\partial r = 0 \longrightarrow$

$\quad \longrightarrow d\dot{r}/dt - r\dot{\phi}^2 = -\mu/r^2 + \mathbb{R}$,

(37) $\quad \mathbb{R} \equiv \partial\, {}^{\epsilon}V/\partial r - d(\partial\, {}^{\epsilon}V/\partial \dot{r})/dt$;

(38) $\quad d(\partial L/\partial \dot{\phi})/dt - \partial L/\partial \phi = 0 \longrightarrow$

$\quad \longrightarrow d(r^2\dot{\phi})/dt = r\mathbb{T}$,

(39) $\quad \mathbb{T} \equiv \partial\, {}^{\epsilon}V/(r\,\partial \phi) - d(\partial\, {}^{\epsilon}V/\partial \dot{\phi})/(r\,dt)$.

We may understand that the disturbing components $\mathbb{R}$ and $\mathbb{T}$ make possible to calculate the variation for the elements which express the solution of the two-body problem. Let us write down here the set of the equations of the variation.

(40) $\quad da/dt = \{2/(n\eta)\}(e \sin f \cdot \mathbb{R} + \xi \cdot \mathbb{T})$,

(41) $\quad de/dt = \{\eta/(nae)\}\{e \sin f \cdot \mathbb{R} + (\xi - \eta^2/\xi) \cdot \mathbb{T}\}$;

(42) $\quad d\varpi/dt = \{\eta/(nae^2)\}\{(1-\xi) \cdot \mathbb{R} + (1 + 1/\xi)\, e \sin f \cdot \mathbb{T}\}$,

(43) $\quad d\varepsilon/dt = [\eta/\{na(1+\eta)\}][\{(1-\xi) - 2\eta(1+\eta)/\xi\} \cdot \mathbb{R} +$
$\quad\quad + (1 + 1/\xi)\, e \sin f \cdot \mathbb{T}]$;

(44) $\quad \xi \equiv p/r = 1 + e \cos f$;

(45) $\quad (p \equiv a\eta^2)$.

§5. Approximated solution for the function ${}^{\epsilon}V$.

In leaving over to solve the partial differential equation (35), we propose here the following expression which shall serve the components of the disturbing force and represent the principal part of *'the energy integral'*.

(46) $\quad {}^{\epsilon}V_1 \equiv {}^{\epsilon}R + d\,{}^{\epsilon}\Gamma(r, \phi; t)/dt$;

where the function ${}^{\epsilon}\Gamma(r, \phi; t)$ shall be served to evaluate an approximated *'energy integral'*. The true longitude ϕ does not exist in the expression (30), ${}^{\epsilon}R$. We will therefore omit this quantity in the following discussion.

(47) $$d\,{}^{\epsilon}\Gamma(r,-;t)/dt = {}^{\epsilon}\Gamma_r \dot{r} + {}^{\epsilon}\Gamma_t \ .$$

The expressions ${}^{\epsilon}\Gamma_r$ and ${}^{\epsilon}\Gamma_t$ stand for respectively the partial derivatives of the function ${}^{\epsilon}\Gamma(r,-;t)$ with respect to r and t.

We are now able to calculate the components of the disturbing force $\mathscr{R}_1$ and $\mathscr{T}_1$ through the approximated disturbing function ${}^{\epsilon}V_1$. The remarkable property of the equations of motion of Lagrange enables us to obtain these components in totally neglecting the existence of the function ${}^{\epsilon}\Gamma(r,-;t)$. This means that we can use the function ${}^{\epsilon}R$ as the disturbing function.

(48) $$\begin{aligned}\mathscr{R}_1 &= \partial\,{}^{\epsilon}V_1/\partial r - d(\partial\,{}^{\epsilon}V_1/\partial \dot{r})/dt = \\ &= \partial\,{}^{\epsilon}R/\partial r - d(\partial\,{}^{\epsilon}R/\partial \dot{r})/dt = \\ &= (\Delta e/e)(\mu/p^2)(\xi^2/e^2)\{(-9/\eta^2+6/\eta^4)\xi^4+ \\ &+(18/\eta^2-12/\eta^4)\xi^3+(-7+23/\eta^2-12/\eta^4)\xi^2+ \\ &+(-22+6/\eta^2)\xi+(1+10\eta^2)-2\eta^4/\xi\} \ ;\end{aligned}$$

(49) $$\begin{aligned}\mathscr{T}_1 &= \partial\,{}^{\epsilon}V_1/(r\,\partial\phi) - d(\partial\,{}^{\epsilon}V_1/\partial\dot{\phi})/(r\,dt) = \\ &= \partial\,{}^{\epsilon}R/(r\,\partial\phi) - d(\partial\,{}^{\epsilon}R/\partial\dot{\phi})/(r\,dt) = \\ &= (\Delta e/e)(\mu/p^2)(\xi^2/e^2)\{(9/\eta^2-6/\eta^4)\xi^3+ \\ &+(6/\eta^2-4/\eta^4)\xi^2+(-2-2/\eta^2)\xi+ \\ &-3\eta^2/\xi+2\eta^4/\xi^2\}\ e\sin f \ .\end{aligned}$$

For these calculations, we utilized the following approximated relations which are found in the solution of the two-body problem.

(50) $$-\mu/(2a) = \{\dot{r}^2 + (r\dot{\phi})^2\}/2 - \mu/r \ ;$$

(51) $$e^2 = 1 - 2r^3\dot{\phi}^2/\mu + r^4\dot{r}^2\dot{\phi}^2/\mu^2 + r^6\dot{\phi}^4/\mu^2 \ .$$

Now we are able to calculate the variation of the elements. As an example, we will write the variation δa_1 for the semi-major axis a.

(52) $$\begin{aligned}\delta a_1 = \int da_1 &= (\Delta e/e)\{2a/(e^2\eta^2)\}\{(-6/\eta^2+4/\eta^4)\xi^4+ \\ &+(3-7/\eta^2+4/\eta^4)\xi^3+(11-3/\eta^2)\xi^2-(1+7\eta^2)\xi\} \ .\end{aligned}$$

For the longitude of the perihelion ϖ and the mean longitude ℓ , we will give the secular parts only.

(53) $$\delta\varpi_{1(s)} = -(\Delta e/e)\{2/(1+\eta)\}\{1/\eta+(e/\eta)^2\}\ f \ .$$

(54) $$\begin{aligned}\delta\ell_{1(s)} &= \int \delta n_1\, dt_{(s)} + \delta\varepsilon_{1(s)} = \\ &= -(\Delta e/e)(9+18\eta+8\eta^2-6\eta^3-7\eta^4)\{e/\eta^2(1+\eta)\}^2\ f \ .\end{aligned}$$

Here, the notation $\delta \ell_{1(s)}$ corresponds to the quantity δn of LeVerrier (1859) or to our $\delta n_{(s)}$ (Inoue, 1992). These secular variations are exactly identical with the results given previously.

We should now evaluate *'the energy integral'* or *the Hamiltonian* H_1, by the relation (33) with the expressions (32) and (46). The result is as follows.

$$(55) \qquad H_1 = (\dot{r}^2 + r^2\dot{\phi}^2)/2 - \{\mu/r + {}^{\epsilon}R\} + $$
$$+\{(\partial\, {}^{\epsilon}R/\partial \dot{r})\cdot\dot{r} + (\partial\, {}^{\epsilon}R/\partial \dot{\phi})\cdot\dot{\phi} - {}^{\epsilon}\Gamma_t\} \ .$$

If the following equality (56) could be satisfied, without any difficulty are we able to obtain the desired *'energy integral'*. Unfortunately, it is impossible to satisfy the equality, because the function ${}^{\epsilon}\Gamma$ is not permitted to include the velocities, namely the elements a and e. Under these circumstances, we will be pleased to obtain this equality only numerically. This sort of treatment is well developed in the investigation of the motion of an artificial satellite with the intermediary orbits (Inoue, 1967). On purpose to clarify this treatment, we will introduce a special mark $\equiv$.

$$(56) \qquad {}^{\epsilon}\Gamma_t \equiv (\partial\, {}^{\epsilon}R/\partial \dot{r})\cdot\dot{r} + (\partial\, {}^{\epsilon}R/\partial \dot{\phi})\cdot\dot{\phi} =$$
$$= (\Delta e/e)(\mu/p)(1/e^2)\{(6/\eta^2 - 4/\eta^4)\xi^4 +$$
$$+ (-2 + 6/\eta^2 - 4/\eta^4)\xi^3 + (-10 + 2/\eta^2)\xi^2 +$$
$$+ (4 + 4\eta^2)\xi - 2\eta^2\} \ .$$

Under the conditions, our *Hamiltonian* H_1 is given as follows.

$$(57) \qquad H_1 = (\dot{r}^2 + r^2\dot{\phi}^2)/2 - \{\mu/r + {}^{\epsilon}R\} \quad ;$$

$$(58) \qquad dH_1/dt = -d\,{}^{\epsilon}\Gamma_t/dt \ .$$

The Hamiltonian H_1 is no longer constant, but changes its value with the time. Nevertheless, the amplitude of this change is very small which is clearly shown through the relation (58) with (56).

§6. Another choice of an approximated solution for the function ${}^{\epsilon}V$.

Suppose the following form for an approximated solution.

$$(59) \qquad {}^{\epsilon}V_2 \equiv \mu\Delta e\,(aB_1/r^2 + B_0/r + B_{-1}/a) \quad ;$$

where the quantities B_1 , B_0 and B_{-1} are functions of e. In substituting this expression in the equation (35), we obtain separated three equations for these unkown quantities. The equations to be solved are given as follows.

$$(60) \qquad dB_1/de + 2eB_1/\eta^2 = 1/2 \quad ,$$

$$(61) \qquad dB_0/de = 1/2 + 1/(2\eta^2) - eB_1/(2\eta^2) \quad ,$$

$$(62) \qquad dB_{-1}/de - 2eB_{-1}/\eta^2 = 1/2 - 1/\eta^2 +$$
$$- eB_1/(2\eta^2) + eB_0/(2\eta^2) \ .$$

We obtained the following solutions.

(63) $B_1 = \quad K_1\eta^2 + (\eta^2/4)\ \log\{(1+e)/(1-e)\}$,

(64) $B_0 = -K_1 e^2/4 + K_0 + 3e/8 +$
$\quad + \{(1+\eta^2/4)/4\}\ \log\{(1+e)/(1-e)\}$,

(65) $B_{-1} = -K_1 e^2\cdot(1-3e^2/8)/(4\eta^2) + K_0 e^2/(4\eta^2) + K_{-1}/\eta^2 +$
$\quad - 49e/(96\eta^2) + 17e/192 +$
$\quad - \{(1-3\eta^2/8)/16\}\ \log\{(1+e)/(1-e)\}$;

where the quantities K_1, K_0 and K_{-1} are the constants of integration.

For the quantity B_{-1}, we have a relation which arises from the term $1/a$.

(66) $dB_{-1}/de - eB_{-1}/(2\eta^2) = -1/(2\eta^2)$.

If we begin with this relation, we can take this as a differential equation for the quantity B_{-1}. But we started from solving the quantity B_1 and after this we solved B_0. We should, therefore, consider the relation (66) as a condition to be satisfied numerically by the quantities K_1, K_0 and K_{-1} as follows.

(67) The left side of (66) with (65) $= - eK_1\{1+15e^4/(32\eta^2)\}/(2\eta^2) +$
$+ eK_0(1-e^2/4)/(2\eta^4) + 3eK_{-1}/(2\eta^4) - 1/(2\eta^4) - 17/(64\eta^4) +$
$+ 11/(128\eta^2) + 23/128 - [15e\{1-8/(15\eta^2)\}/256]\ \log\{(1+e)/(1-e)\} =$

$\equiv$ The right side of (66) $= - 1/(2\eta^2)$.

In order to make much clear what we mean, we write the condition (67) in giving a numerical value 0.205 6105 for the eccentricity e.

(68) $- 0.107\,4371K_1 + 0.110\,8970K_0 + 0.336\,2447K_{-1} - 0.567\,5175 =$
$\quad = - 0.522\,0709$.

The quantities K_1, K_0 and K_{-1} are not the dynamical ones. They are merely the constants of integration, but which can be served to adjust the numerical values in given relations. The partial differential equation (35) is satisfied under the following form with the special mark $\equiv$.

(69) $\epsilon R \equiv \epsilon V_2 - (\partial\epsilon V_2/\partial\dot{r})\cdot\dot{r} - (\epsilon V_2/\partial\dot{\phi})\cdot\dot{\phi}$.

Even though it is an approximated one, we can now calculate the compontents $\mathbb{R}_2$ and $\mathbb{T}_2$ of the disturbing force as follows.

(70) $\mathbb{R}_2 = (\mu\Delta e/p^2)\,[-(1/e\eta^2)\xi^5 + [5/(2e\eta^2) - 1/(2e) +$
$\quad + (1/8)\ \log\{(1+e)/(1-e)\} + K_1/2]\,\xi^4 + (2/e\eta^2)\,\xi^3 +$
$\quad + [-4/e + e/2 + (\eta^2/8)\ \log\{(1+e)/(1-e)\} + \eta^2 K_1/2]\,\xi^2 + C^*\xi]$,

(71) $\mathbb{T}_2 = (\mu\Delta e/p^2)\ e\ \sin\ f\ [(1/e\eta^2)\xi^4 + [1/(2e\eta^2) + 1/(2e) +$
$\quad - (1/8)\ \log\{(1+e)/(1-e)\} - K_1/2]\,\xi^3 +$
$\quad + [1/e - e/2 - (\eta^2/8)\ \log\{(1+e)/(1-e)\} - \eta^2 K_1/2]\,\xi - C^*\ ]$;

(72) $C^* \equiv 1/e + 21e/16 - 13e^3/48 + (3\eta^4/32)\ \log\{(1+e)/(1-e)\} +$
$\quad + K_1(1-3e^2/4+3e^4/8) - K_0 - 4K_{-1}$.

We will give below the secular parts of the longitude of the perihelion ϖ and of the mean longitude ℓ.

$$(73)\quad \delta\varpi_{2(s)} = (\Delta e/e)[(1+\eta+\eta^2)/(2\eta)+ -(e/4)(1+\eta/2)\log\{(1+e)/(1-e)\}+ -K_1 e(1+\eta/2)-C^* e/(2\eta^3)]\ f \quad ;$$

$$(74)\quad \delta\ell_{2(s)} = (\Delta e/e)[1/(2\eta)-2+\eta^2/2+ -(e/4)(1+\eta/2+\eta^2/2)\log\{(1+e)/(1-e)\}+ -K_1 e(1+\eta/2+\eta^2/2)+C^* e/(2\eta^3)(5\eta-1)]\ f \quad .$$

If we suppose the existence of the following equalities for the quantities $\delta\varpi_{1(s)}$, $\delta\varpi_{2(s)}$; $\delta\ell_{1(s)}$, $\delta\ell_{2(s)}$ as well as the equality (67) or (69), we will obtain the sufficient equations in order to solve with respect to the three constants of integration K_1 , K_0 and K_{-1} .

$$(75)\qquad \delta\varpi_{1(s)} \equiv \delta\varpi_{2(s)} \quad ,$$

$$(76)\qquad \delta\ell_{1(s)} \equiv \delta\ell_{2(s)} \quad .$$

It is not necessary to give here concretely the expressions for these constants of integration.

§7. Contemporary significance of the discussions about *LeVerrier*.

There might be someone who criticizes as below " *LeVerrier ? Well, he is a man who made discover the planet Neptune and certainly is he a great astronomer. But it's all over with him.* " Anybody knows that LeVerrier is the man who pointed out first the excess advance in the longitude of the perihelion. His theory and the observational data he used are in fact antiquated. However, the application of his theory to the observational data used by him gave us the following result (Inoue, 1993). The expression e' indicates the secular variation of e.

$$(77)\qquad \varpi' = 39.''8\,129/\text{century} - 2.725\ e' \quad ,$$

$$(78)\qquad \delta n = 10.''5\,087/\text{century} - 0.1991\ e' \quad .$$

His result is as follows.

$$(79)\qquad \varpi' = 38.''3\quad /\text{century} \quad ,$$

$$(80)\qquad \delta n = 10.''3\,9\quad /\text{century} \quad .$$

For these calculations, there exists a tiny difference between his procedure and ours. That is, he analyzed the transit observations of May and those of November separately and we achieved the same analysis in unifying the whole observations. The essence of the procedure for the analysis of the excess advance problem does remain invariably since the LeVerrier's epoch. Either way, nobody knows that the LeVerrier's result is denied in the frame of the Newtonian mechanics. This means that it is still significant to investigate about *LeVerrier*.

We will give here the numerical values for $\delta\varpi_{1(s)}$, (53) and $\delta\ell_{1(s)}$, (54) with Δe, (31) which is evaluated by Δe_1, (18) and Δe_2, (19).

(81) $\delta \varpi_{1\,(s)} = -43.''3\ 7489/\text{century}$,

(82) $\delta \Omega_{1\,(s)} = -10.''5\ 3667/\text{century}$;

(83) $\Delta e = +\ 3.''17\ 2675\times 10^{-3} = 1.53\ 8156\times 10^{-8}$.

We took here much more figures than the realistic ones. It is merely because of the reason to show what we treat and calculate.

Tenacious and systematic examinations about *LeVerrier* enabled us to reach the idea of the intermediary orbits and to terminate thoroughly the problem of the excess advance motion in the longitude of the perihelion of Mercury. We were able to be confident in our justice through detailed representations of Brumberg (1991) and the application of the post-Keplerian orbit for the timing of binary pulsars (Kopeikin, 1994).

Roseveare (1982) enumerated the possibilities for the solution of the problem as follows.

① Unknown causes such as the existence of intra-Mercurial objects and the improvement of the planetary masses.
② Modification of the law of gravitation.
③ Neglect of the existence of these discrepancies.

And then, he gave the following fourth possibility.

> *A fourth possibility was also available —— that a mistake had been made in the theoretical work and no such anomalous motion really occured. But this was difficult to test because the advanced nature of the work meant few people could tackle it and it would take a long time if they did.*

We would like to declare that we did this.

§8. Conclusion.

We may summarize our results as follows.

(α) LeVerrier obtained correctly the value for the advance in the longitude of the perihelion due to the planets, namely 526.″7/century.

(β) The traditional treatment for the perturbation due to Venus caused the intrusion of an intermediary orbit, in other words, an ellipse which is retrograding by 43″/century.

(γ) Our comprehension for the appearance of the discrepancy 43″/century is simply illustrated as follows.

(Observation)		(Calculation)		(Discrepancy)
527″/ century	−	{527″/ century+(−43″/ century)}	=	+43″/ century

After all, the theory of the motion of Mercury was completely established by LeVerrier about 150 years ago.

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