水星近日点前進問題の解決

Termination of the problem of the excess advance in the longitude of the perihelion of Mercury

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Abstract. Since Le Verrier, who proposed the problem of the excess advance in the longitude of the perihelion of Mercury, one took this problem real. The present study will make clear that an inadequate treatment, brought on this advance problem in the theory of planetary motion. When one attempts to include in the elliptic orbit a part of planetary perturbations, there appears a small quantity concerning a correction for the eccentricity. Even though this quantity is extremely small, one should retain this in the perturbation theory. Le Verrier and others did not recognize this fact, and then neglected totally the quantity. Because of this, one admits the apparition of the excess advance.

1. Terms containing the mean anomaly of Mercury.

In the theory of motion of Mercury, there exist terms which depend uniquely on the mean anomaly of Mercury itself, say “M”. Under the circumstances, it is not easy to separate the perturbed parts from the non-perturbed elliptic parts when one makes an effort at the numerical evaluation of the elliptic elements by means of the observational data.

Le Verrier (1856) paid a special attention to this point. Let’s express these terms under the following forms in order to see the essence of his idea [page 39].

(1) $$\delta\phi = S\sin M + C\cos M,$$

(2) $$\delta r = S'\sin M + C'\cos M;$$

where $\delta\phi$ and δr are respectively the planetary perturbations on the true longitude ϕ and the radius vector r.

There will appear analogous terms, if one changes slightly the values of the eccentricity e and the longitude of the perihelion ϖ. Indeed, if one puts these changes Δe and $\Delta\varpi$, then one obtains these terms through the following formulae：

(3) $$\Delta\phi = 2\Delta e\sin M - 2e\Delta\varpi\cos M,$$

(4) $$\Delta r = -ae\Delta\varpi\sin M - a\Delta e\cos M.$$

In summing up, one comes to hand the following expressions :

(5) $$\delta\phi + \Delta\phi = (S + 2\Delta e)\sin M + (C - 2e\Delta\varpi)\cos M,$$

(6) $$\delta r + \Delta r = (S' - ae\Delta\varpi)\sin M + (C' - a\Delta e)\cos M.$$

If one choses the values Δe and $\Delta\varpi$ as follows :

(7) $$S + 2\Delta e = 0 \quad , \quad C - 2e\Delta\varpi = 0 \quad ;$$

one obtains the following expressions as modified perturbations for the true longitude and the radius vector [page 40]:

(8) $$d(\phi) \equiv \delta\phi + \Delta\phi = 0 \quad ,$$

(9) $$d(r) \equiv \delta r + \Delta r = (S' - aC/2)\varpi \sin M + (C' + aS/2)\cos M.$$

Up to this, his treatment is well legitimate for an approximate theory.

2. **Action due to Venus.**

When Le Verrier constructed the theory of motion of Mercury, he did not treat as mentioned above (1859). Because the action caused by Venus on the radius vector of Mercury is sufficiently small, he supposed the existence of the following equalities for the two quantities Δe and $\Delta\varpi$:

(10) $$S + 2\Delta e = 0 \quad , \quad C - 2e\Delta\varpi = 0 \quad ;$$

(11) $$S' - ae\Delta\varpi = 0 \quad , \quad C' - a\Delta e = 0 \quad .$$

In order to make clear what he did, we would like to cite a part of the sentences from his reports [page 16] :

> Il résulte de ces considérations que les termes des perturbations qui dépendent uniquement de la longitude de Mercure peuvent être négligés soit dans **la longitude**, soit dans **le rayon de Mercure**.

This means that he considered it is possible to determine two values of correction for each element. One value should not be in general equal to the other for each of these elements. There might surely exist a discrepancy between these two values. Even in a case where the discrepancy is extremely small, one should pay a very careful attention especially to the case of the eccentricity.

On purpose to clarify what we want to say, let us introduce two quantities Δe_M and Δe_r through the following relations :

(12) $$\Delta e_M \equiv -S/2 \quad , \quad \Delta e_r \equiv C'/a \quad .$$

For the numerical evaluation of Δe_M and Δe_r, we would like to use the values given by Le Verrier [page 13] :

(13) $$\delta\phi = +0.''017 \sin\lambda - 0.''063 \cos\lambda \quad ,$$

(14) $$\delta r = -0.''011 \sin\lambda - 0.''003 \cos\lambda \quad ;$$

here the quantity λ stands for the mean longitude which equals to $\varpi + M$. In order to convert from λ to M, we will apply the numerical value for ϖ, 75°07' [page 21] :

(15) $$\delta\phi = +0.''065252 \sin M + 0.''000248 \cos M \quad ,$$

(16) $$\delta r = +0.''000074 \sin M - 0.''011401 \cos M \quad .$$

With the value of a : 0.3870987 [page 23], we obtain as follows :

(17) $$\Delta e_M = -0.''032626 \quad , \quad [\, d(\phi) = 0 \,] \quad ,$$

(18) $$\Delta e_r = -0.''029453 \quad , \quad [\, d(r) = 0 \,] \quad .$$

The discrepancy between Δe_M and Δe_r is indeed extremely small.

(19) $$\Delta e \equiv \Delta e_r - \Delta e_M = +0.''003173 = 1.5381 \times 10^{-8}$$

Le Verrier neglected this discrepancy and believed that his starting orbit is always the solution of the two-body problem. This is undoubtedly **the very source** of the troublesome advance problem. **Up to today, nobody recognized this fact**.

3. Intermediary orbit.

Le Verrier had neglected the discrepancy, but he calculated for the ephemerides up to much more higher orders [page 12]. We would like therefore to retain this discrepancy in what follows and introduce the two quantities e_M and e_r :

(20) $$e_M \equiv e + \Delta e_M \quad , \quad e_r \equiv e + \Delta e_r \quad .$$

We consider an orbit which has two eccentricities, if we say by analogy with the elliptic orbit.

For the quantity e_M, we would like to apply the numerical value 0.2056105 which is given for e by Le Verrier [page 21].

The orbit should not be an ellipse but a slightly different one. We take this as an intermediary orbit (Garfinkel, 1958), and consider a planar motion which is defined by the following formulae :

$$(21) \qquad M = \int_{t_0}^{t} n dt + \chi \quad , \quad n \equiv \sqrt{\mu / a^3} \quad , \qquad \mu \equiv G(m_0 + m) \quad ;$$

$$(22) \qquad u - e_M \sin u = M \quad ;$$

$$(23) \qquad \tan \frac{f}{2} = \sqrt{\frac{1+e_r}{1-e_r}} \tan \frac{u}{2} \quad ;$$

$$(24) \qquad \phi = \varpi + f \quad ,$$

$$(25) \qquad r = a(1 - e_r \cos u) \quad .$$

This is an orbit including the four dynamical constants, namely $a, e_r ; \varpi, \chi$ which we call the elements on the analogy of the elliptic one, because the physical meaning is wholly analogous. The quantities u and f are respectively analogous ones to the eccentric anomaly and the true anomaly in the elliptic orbit.

This orbit has the energy integral. But, it is not easy to find the precise form. We are then obliged to give here merely an approximate form which is, nevertheless, sufficient for the present study. If we write E for the energy constant, we have the following form of an approximated integral :

$$(26) \qquad E = -\frac{\mu}{2a} = \frac{1}{2}\left\{\dot{r}^2 + (r\dot{\phi})^2\right\} - \left\{\frac{\mu}{r} + \varepsilon R\right\} \quad ;$$

$$(27) \qquad \varepsilon R \equiv \mu \frac{\Delta e}{e_r}\left\{-\frac{2a}{r^2} + \frac{3}{r} - \frac{1}{a}\right\} \quad .$$

Through this expression, we recognize that the existence of Δe makes different the intermediary orbit from the elliptic orbit. Le Verrier regarded the orbit as an ellipse in putting $\Delta e = 0$.

4. Variation of the constants.

Being able to consider the dynamical constants : $a, e_r ; \varpi, \chi$ as the contact elements, we can apply the method of the variation of the arbitrary constants (Brumberg, 1991). The quantity εR plays the role of the disturbing function for the elements. The equations of the variation for these elements are the same as the ordinary ones. If one feels some inquietudes because of the existence of the constants a and e_r in the function εR, it is desirable to refer to the study of Garfinkel and Aksnes (1970). There is no secular variation in the elements a and e_r. The elements ϖ and χ have the secular variations as follows :

(28) $$\delta\varpi_{(s)} = \frac{\Delta e}{e_r}\left\{\frac{-2-2\eta+2\eta^2}{\eta^2(1+\eta)}\right\}f \quad ,$$

(29) $$\delta n_{(s)} = \frac{\Delta e}{e_r}\left[\frac{-9-18\eta-8\eta^2+6\eta^3+7\eta^4}{\{\eta^2(1+\eta)\}^2}\times e_r^2\right]f \quad ;$$

(30) $$\eta \equiv \sqrt{1-e_r^2} \quad ,$$

where $\delta n_{(s)}$ expresses the secular part of the mean longitude λ. Being rather an inadequate notation, Le Verrier used such one.

We can immediately calculate the numerical values for these secular variations :

(31) $$\delta\varpi_{(s)} = -43.''374/century \quad ,$$

(32) $$\delta n_{(s)} = -10.''536/century \quad .$$

The result shows that our intermediary orbit incorporates a retrograde secular variation in the longitude of the perihelion of a magnitude of some 40 arc-seconds per century. Le Verrier's ephemerides contain the corrections given by (17) and (18), but not the difference of these two given in (19) [pages 22 and 23]. This is the reason why nobody could explain in the frame of the Newtonian mechanics the discrepancy existing between the transit observations and the theory of motion of Mercury.

We will cite the results obtained by Le Verrier in order to compare our present results with his [page 96] :

(33) $$\varpi' = +38.''3/century \quad ,$$

(34) $$\delta n = +10.''39/century \quad .$$

Our $\delta n_{(s)}$ has a close value to Le Verrier's δn, if one changes its sign. Newcomb gave, for ϖ', a value of 43 arc-seconds per century (Chebotarev, 1967), which is also very close to $-\delta\varpi_{(s)}$. Newcomb succeeded the method of Le Verrier for the construction of the planetary theories (Clemence, 1943). The similar advance difficulty does therefore appear in his case also.

Conclusion. The present study made clear that one should retain the small quantity when one constructs the theory of planetary motion. This quantity introduces an intermediary orbit, instead of the ordinary elliptic orbit, which incorporates strictly the same, but the opposite sign, amount of the excess advance in the longitude of the perihelion of Mercury. Besides this, an excess variation existing in the mean longitude is also correctly explained through the present investigations. (93127ME2139)

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Dans les présentes études, il y existe une erreur qui était déjà bien corrigée (Inoue,T. 1994 : Proceedings of the Twenty-Sixth Symposium on Celestial Mechanics). Je voudrais ici reproduire ce que j'y ai fait.

Dans la relation (23), la quantité e_r doit être remplacée par e_M . C'est ainsi qu'il y aura un changement dans l'expression (27) comme il suit :

$$\varepsilon R \equiv \mu \frac{\Delta e}{e}\left\{-\frac{(a\eta)^2}{r^3}-\frac{a}{r^2}+\frac{3}{r}-\frac{1}{a}\right\}$$

Ce changement influence la quantité $\delta n_{(s)}$ dont l'expression doit se modifier. Quant à la quantité $\delta\varpi_{(s)}$, il n'y a aucun changement. *Takeshi INOUE*